

Partially Magic Labelings and the Antimagic Graph Conjecture

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Abstract. The *Antimagic Graph Conjecture* asserts that every connected graph $G = (V, E)$ except K_2 admits an edge labeling such that each label $1, 2, \dots, |E|$ is used exactly once and the sums of the labels on all edges incident to a given vertex are distinct. On the other extreme, an edge labeling is *magic* if the sums of the labels on all edges incident to each vertex are the same. In this paper we approach antimagic labelings by introducing *partially magic labelings*, where “magic occurs” just in a subset of V . We generalize Stanley’s theorem about the magic graph labeling counting function to the associated counting function of partially magic labelings and prove that it is a quasi-polynomial of period at most 2. This allows us to introduce *weak antimagic labelings* (for which repetition is allowed), and we show that every bipartite graph satisfies a weakened version of the Antimagic Graph Conjecture.

Résumé. La Conjecture des Graphes Antimagiques stipule que tout graphe connexe $G = (V, E)$, à l’exception de K_2 , admet un étiquetage de bord de sorte que chaque étiquette $1, 2, \dots, |E|$ est utilisée exactement une fois, et que les sommes des étiquettes sur toutes les arêtes incidentes à un sommet donné sont distinctes. À l’opposé, un étiquetage de bord est magique si les sommes des étiquettes sur toutes les arêtes incidentes à chaque sommet sont les mêmes. Dans cet article, nous nous attaquons aux marquages antimagiques en introduisant le concept de marquages partiellement magiques, pour lesquels “la magie ne se produit” que dans un sous-ensemble de V . Nous généralisons le théorème de Stanley sur la fonction de comptage d’étiquetage de graphe de magie au cas des marquages partiellement magiques, et nous prouvons que cette fonction est quasi-polynomiale de période au plus 2. Cela nous permet d’introduire l’idée des marquages faiblement antimagiques (pour lesquelles la répétition est autorisée), et nous montrons que tout graphe bipartite satisfait une version affaiblie de la Conjecture des Graphes Antimagiques.

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1 Introduction

Graph theory is abundant with fascinating open problems. In this paper we propose a new *ansatz* to a long-standing and still-wide-open conjecture, the *Antimagic Graph Conjecture*. Our approach generalizes Stanley’s enumeration results for magic labelings of a graph [14] to partially magic labelings, with which we analyze the structure of antimagic labelings of graphs.

Let G be a finite graph, which may have loops and multiple edges. We shall denote the set of vertices of G by V and the set of edges by E . A **labeling** of G is an assignment $L : E \rightarrow \mathbb{Z}_{\geq 0}$ of a nonnegative integer $L(e)$ to each edge e of G and a k -**labeling** is one where each edge label is among $0, 1, \dots, k$. If for every vertex v of G the sum $s(v)$ of the labels of all edges incident to v equals r (counting each loop at v only once) then L is called a **magic labeling of G of index r** . In the 1970s, Stanley proved some remarkable facts for magic labelings:

Theorem 1 (Stanley [14]). *Let G be a finite graph and define $H_G(r)$ to be the number of magic labelings of G of index r . There exist polynomials $P_G(r)$ and $Q_G(r)$ such that $H_G(r) = P_G(r) + (-1)^r Q_G(r)$. Moreover, if the graph obtained by removing all loops from G is bipartite, then $Q_G(r) = 0$, i.e., $H_G(r)$ is a polynomial of r .*

This theorem can be rephrased in the language of quasi-polynomials. Recall that a **quasi-polynomial** is a function $f : \mathbb{Z} \rightarrow \mathbb{C}$ of the form $f(n) = c_n(k)k^n + \dots + c_1(k)k + c_0(k)$ where $c_0(k), \dots, c_n(k)$ are periodic functions in k and the **period** of f is the least common multiple of the periods of $c_0(k), \dots, c_n(k)$. **Theorem 1** says that $H_G(r)$ is a quasi-polynomial of period at most 2.

On the other extreme, a labeling is **antimagic** if each edge label is a distinct element of $\{1, \dots, |E|\}$ so that the sums $s(v)$ are distinct. It has been conjectured for more than two decades that K_2 is essentially the only graph for which we cannot find an antimagic labeling [11]:

Antimagic Graph Conjecture. *Every connected graph except for K_2 admits an antimagic labeling.*

Surprisingly this conjecture is still open for **trees**, i.e., connected graphs without cycles, though it has been proven that trees without vertices of degree 2 admit an antimagic labeling [13]. Moreover, the validity of the Antimagic Graph Conjecture was proved in [1] for connected graphs with minimum degree $\geq c \log |V|$ (where c is a universal constant), for connected graphs with maximum degree $\geq |V| - 2$, and in [9] for graphs with average degree at least a universal constant. We also know that connected k -regular graphs with $k \geq 2$ are antimagic [8, 5, 6]. Furthermore, all Cartesian products of regular graphs of positive degree are antimagic [7], as are joins of complete graphs [2]. For more related results, see the comprehensive survey [10] on graph labelings.

In the classic definition of antimagic labelings, labels are distinct, however, for magic labelings repetition is allowed. Borrowing a leaf from the latter, we soften the requirement in the antimagic definition above and say a labeling is **weakly antimagic** if each edge label is an element of $\{1, \dots, |E|\}$ so that the sums $s(v)$ are distinct. In other words, we allow repetition among the labels. Our first main result is as follows.

Theorem 2. *Let G be a finite graph. Then the number $A_G(k)$ of weakly antimagic k -labelings is a quasi-polynomial in k of period at most 2. Moreover if the graph G minus its loops is bipartite, then $A_G(k)$ is a polynomial in k .*

We remark that antimagic counting functions of the flavor of $A_G(k)$ already surfaced in [4]. At any rate, **Theorem 2** implies that for bipartite graphs we have a chance of using the polynomial structure of $A_G(k)$ to say something about the antimagic character of G . Namely, the number of k -labelings is at most $(k+1)^{|E|}$ and so the degree of the polynomial $A_G(k)$ is at most $|E|$. Therefore, $A_G(k)$ can have at most $|E|$ integer roots, one of which is 0, as we will show below. Also, by definition we know that $A_G(k+1) \geq A_G(k)$ for any $k \in \mathbb{Z}_{\geq 0}$, and so $A_G(|E|)$ cannot be zero. What we just illustrated (though we will give a careful proof below) is that **Theorem 2** implies:

Theorem 3. *Every bipartite graph without a K_2 component admits a weakly antimagic labeling. Furthermore, every graph $G = (V, E)$ without a K_2 component admits a labeling with distinct vertex sums $s(v)$ and labels in $\{1, 2, \dots, 2|E|\}$.*

We approach weakly antimagic labelings by introducing a new twist on magic labelings which might be of independent interest. Fix a subset S of vertices of G . A **partially magic labeling of G over S** is a labeling such that “magic occurs” just in S , that is, for all $v \in S$ the sums $s(v)$ are equal.

Theorem 4. *Let G be a finite graph and $S \subseteq V$. The number $M_S(k)$ of partially magic k -labelings over S is a quasi-polynomial in k with period at most 2. Moreover, if the graph G minus its loops is bipartite, then $M_S(k)$ is a polynomial in k .*

In order to prove **Theorem 4**, we will follow Stanley’s lead in [14] and use linear Diophantine homogeneous equation and Ehrhart quasi-polynomials to describe partially magic labelings of the graph G ; **Section 2** contains a proof of **Theorem 4**. In **Section 3** we prove **Theorems 2** and **3**. We conclude in **Section 4** with some comments on a directed version of the Antimagic Graph Conjecture, as well as open problems.

2 Enumerating Partially Magic Labeling

Given a finite graph $G = (V, E)$ and a subset $S \subseteq V$, we introduce an indeterminate z_e for each edge e and let $\{v_1, \dots, v_s\}$ be the set of all vertices of S , where $|S| = s$. In

this setup, a partially magic k -labeling over S corresponds to an integer solution of the system of equations and inequalities

$$\sum_{e \text{ incident to } v_j} z_e = \sum_{e \text{ incident to } v_{j+1}} z_e \quad (1 \leq j \leq s-1) \quad \text{and} \quad 0 \leq z_e \leq k. \quad (2.1)$$

Define Φ as the set of all pairs (L, k) where $L \in \mathbb{Z}_{\geq 0}^E$ is a partially magic k -labeling; that is, (L, k) is a solution to (2.1). If L is a partially magic k -labeling and L' is a partially magic k' -labeling, then $L + L'$ is a partially magic $(k + k')$ -labeling. Thus Φ is a semigroup with identity $\mathbf{0}$. (This is also evident from (2.1).)

For the next step, we will use the language of generating functions, encoding all partially magic k -labelings as monomials. Let $q = |E|$ and define

$$F(Z) = F(z_1, \dots, z_q, z_{q+1}) := \sum_{(L, k) \in \Phi} z_1^{L(e_1)} \cdots z_q^{L(e_q)} z_{q+1}^k. \quad (2.2)$$

Note that if we substitute $z_1 = \cdots = z_q = 1$ in $F(Z)$, we enumerate all partially magic k -labelings:

$$F(\mathbf{1}, z_{q+1}) = \sum_{(L, k) \in \Phi} z_{q+1}^k = \sum_{k \geq 0} M_S(k) z_{q+1}^k, \quad (2.3)$$

where we abbreviated $\mathbf{1} := (1, 1, \dots, 1)$.

We call a nonzero element $\alpha = (\alpha_1, \dots, \alpha_q, k) \in \Phi$ **fundamental** if it cannot be written as the sum of two nonzero elements of Φ ; furthermore, α is **completely fundamental** if no positive integer multiple of it can be written as the sum of nonzero, nonparallel elements of Φ (i.e., they are not scalar multiple of each other). In other words, a completely fundamental element $\alpha \in \Phi$ is a nonnegative integer vector such that for each positive integer n , if $n\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Phi$, then $\beta = j\alpha$ and $\gamma = (n - j)\alpha$ for some nonnegative integer j . Note that by taking $n = 1$ in the above definition, we see that every completely fundamental element is fundamental. Also note that any fundamental element $(\alpha_1, \dots, \alpha_q, k)$ necessarily satisfies $k = \max\{\alpha_1, \dots, \alpha_q\}$.

Now we focus on the generating functions (2.2) and (2.3) and employ [15, Thm. 4.5.11], which says in our case that the generation function $F(Z)$ can be written as a rational function with denominator

$$D(Z) := \prod_{\alpha \in \text{CF}(\Phi)} (1 - Z^\alpha), \quad (2.4)$$

where $\text{CF}(\Phi)$ is the set of completely fundamental elements of Φ and we used the monomial notation $Z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_q^{\alpha_q} z_{q+1}^k$. To make use of (2.4), we need to know some information about completely fundamental solutions to (2.1). To this extent, we borrow the following lemmas from magic labelings [14], i.e., the case $S = V$:

Lemma 5. For a finite graph G , every completely fundamental magic labeling has index 1 or 2. More precisely, if L is any magic labeling of G , then $2L$ is a sum of magic labelings of index 2.

Lemma 6. For a finite graph G , the following conditions are equivalent:

- (1) Every completely fundamental magic labeling of G has index 1.
- (2) If G' is any spanning subgraph of G such that every connected component of G' is a loop, an edge, or a cycle of length ≥ 3 , then every one of these cycles of length greater than or equal to 3 must have even length.

Lemma 5 implies that every completely fundamental magic labelings has index 1 or 2 and therefore, it cannot have a label ≥ 3 (because labels are nonnegative). By the same reasoning, if G satisfies the condition (2) in **Lemma 6**, every completely fundamental magic labeling of it has index 1 and so cannot have labels ≥ 2 . We now give the analogous result for *partially* magic labelings:

Lemma 7. Every completely fundamental partially magic labeling of G over S has labels 0, 1, or 2.

Proof. If $S = V$, then every completely fundamental partially magic labeling over S is a completely fundamental magic labeling over G . By **Lemma 5**, it has index 1 or 2 and so the labels are among 0, 1, or 2.

Suppose that $S \subsetneq V$ and let L be a partially magic labeling of G over S that has a label ≥ 3 on the edge e which is incident to vertices u and v . We will show that L is not completely fundamental. There are three cases:

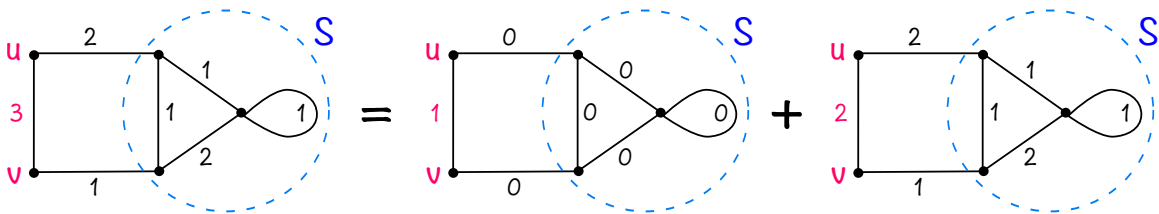


Figure 1: A non-completely fundamental partially magic labeling in Case 1.

Case 1: $u, v \notin S$, that is, e is not incident to any vertex in S . We can write L as the sum of L' and L'' , where all the labels of L' are zero except for e with $L'(e) = 1$ and all the labels of L'' are the same as L except e with $L''(e) = L(e) - 1$; see **Figure 1**. Since L' and L'' are both partially magic over S , then by definition L is not a completely fundamental partially magic labeling over S .

Case 2: $u \notin S$ and $v \in S$. Let G_S be the graph with vertex set S obtained from G by removing all the edges of G that are not incident to some vertex of S and making loops

out of those edges that are incident to both S and $V \setminus S$. Now define a labeling L_S over G_S such that all the edges that are incident to S get the same labels as L and all the new loops get the labels of L that were on the original edges, as in **Figure 2**.

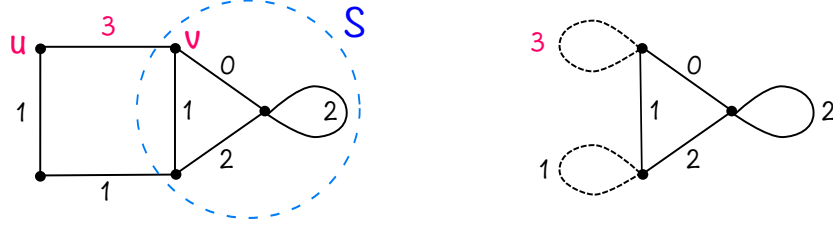


Figure 2: A graph G_S and magic labeling L_S in Case 2.

Since L is partially magic over S , L_S is a magic labeling of G_S . However, $L_S(e) = L(e) \geq 3$ and so S has a vertex with sum ≥ 3 . Therefore, by **Lemma 5**, L_S is not a completely fundamental magic labeling of G_S and so there exist magic labelings L_S^i of index 2 such that $2L_S = \sum L_S^i$, as in **Figure 3**. Now we extend each magic labeling L_S^i to

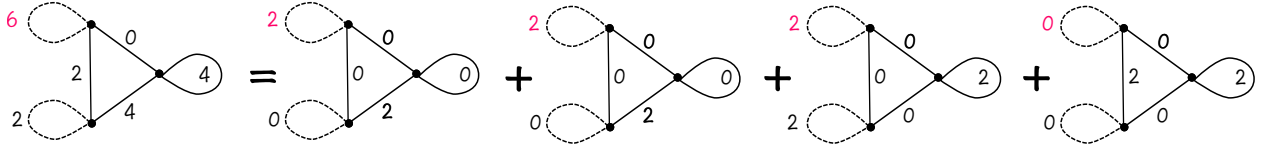


Figure 3: A graph G_S and the magic labelings L_S^i where $2L_S = \sum_{i=0}^4 L_S^i$.

a partially magic labeling L^i over G as follows, for $i = 1, 2$ we define:

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or} \\ & e \text{ is incident to both vertices of } S \text{ and } V \setminus S, \\ L(e) & \text{if } e \text{ is not incident to } S. \end{cases}$$

For $i \geq 3$, the extensions are:

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or} \\ & e \text{ is incident to both vertices of } S \text{ and } V \setminus S, \\ 0 & \text{if } e \text{ is not incident to } S. \end{cases}$$

Therefore $2L(e) = \sum L^i(e)$ for all $e \in E$; see **Figure 4**. By definition, L^i is nonzero partially magic labeling of G over S with labels among $0, 1, 2$, for every $i > 1$. This proves that L is not a completely fundamental partially magic labeling.

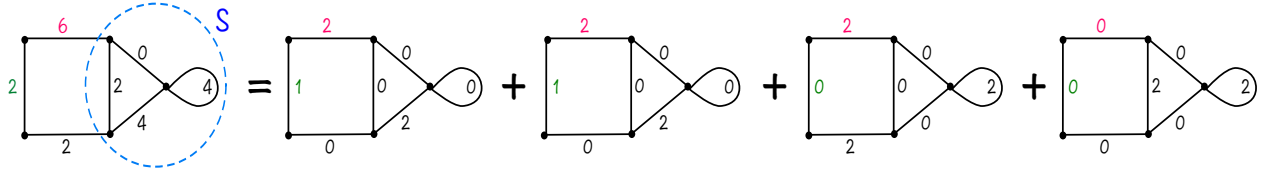


Figure 4: A non-completely fundamental partially magic labeling L with $2L = \sum_{i=0}^4 L^i$.

Case 3: $u, v \in S$. The argument is similar to Case 2, by constructing the graph G_S with the labeling L_S . Since L is partially magic labeling over S , L_S is a magic labeling of G_S . However, it is not completely fundamental because it has a loop e with label $L_S(e) = L(e) \geq 3$. So there exist magic labelings L_S^i of index 2, such that $2L_S = \sum L_S^i$. We extend each L_S^i to a labeling L^i over G as follows, for $i = 1, 2$ we define:

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or} \\ & e \text{ is incident to both vertices of } S \text{ and } V \setminus S, \\ L(e) & \text{if } e \text{ is not incident to } S. \end{cases}$$

For $i \geq 3$, we extend the labeling L_S^i to L^i over G as follows:

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or} \\ & e \text{ is incident to both vertices of } S \text{ and } V \setminus S, \\ 0 & \text{if } e \text{ is not incident to } S. \end{cases}$$

By definition, $2L = \sum L^i$ where each L^i is a partially magic labeling over S and has labels 0, 1, or 2. Therefore, L is not a completely fundamental magic labeling of G over S . \square

Proof of Theorem 4. By (2.3) and (2.4),

$$F(\mathbf{1}, z) = \sum_{k \geq 0} M_S(k) z^k$$

is a rational function with denominator

$$D(\mathbf{1}, z) = \prod_{\beta \in \text{CF}(\Phi)} (1 - \mathbf{1}^\beta z^k) \quad (2.5)$$

where $\text{CF}(\Phi)$ is the set of completely fundamental elements of Φ . According to Lemma 7, every completely fundamental element of Φ has labels at most 2. Therefore

$$\sum_{k \geq 0} M_S(k) z^k = \frac{h(z)}{(1-z)^a (1-z^2)^b} \quad (2.6)$$

for some nonnegative integers a and b , and some polynomial $h(z)$. Basic results on rational generating functions (see, e.g., [15]) imply that $M_S(k)$ is a quasi-polynomial in k with period at most 2.

Now let G be a bipartite graph and $S \subseteq V$. We know that all the cycles of G have even length, so it satisfies condition (2) in Lemma 6. Therefore every completely fundamental magic labeling of G has index 1 and so it cannot have a label ≥ 2 . For partially magic labelings of G , we can use the same procedure of Lemma 7 to see that if L is a completely fundamental partially magic labeling of G , it cannot have a label ≥ 2 . Therefore, in the generating function (2.6), we have $b = 0$ and so $M_S(k)$ is a polynomial in k . \square

The equations in (2.1) together with $z_e \geq 0$ describe a pointed rational cone, and adding the inequalities $z_e \leq 1$ gives a rational polytope \mathcal{P}_S . Our reason for considering \mathcal{P}_S are structural results, due to Ehrhart and Macdonald (see, e.g., [3]), about the lattice-point enumerator of any polytope $\mathcal{P} \subset \mathbb{R}^d$,

$$L_{\mathcal{P}}(t) := \left| t\mathcal{P} \cap \mathbb{Z}^d \right|,$$

where t is a positive integer and $t\mathcal{P} := \{tx : x \in \mathcal{P}\}$ denotes the t^{th} dilate of \mathcal{P} . A partially magic k -labeling of a graph G with labels among $\{0, 1, \dots, k\}$ (which is a solution of (2.1)) is therefore an integer lattice point in the k -dilation of \mathcal{P}_S , i.e.,

$$M_S(k) = L_{\mathcal{P}_S}(k).$$

Let $M_S^{\circ}(k)$ be the number of *positive* partially magic labelings of a graph G over a subset S of vertices of G , that is, a partially magic labeling with labels among $\{1, \dots, k\}$. Thus $M_S^{\circ}(k) = L_{\mathcal{P}_S^{\circ}}(k+1)$, where \mathcal{P}_S° is the relative interior of the polytope \mathcal{P}_S . Ehrhart's famous theorem implies that $L_{\mathcal{P}_S}(t)$ is a quasi-polynomial in t of degree $\dim \mathcal{P}_S$, and the Ehrhart–Macdonald reciprocity theorem for rational polytopes gives the algebraic relation $(-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-t) = L_{\mathcal{P}^{\circ}}(t)$, which implies for us:

Corollary 8. *Let $G = (V, E)$ be a graph and $S \subseteq V$. Then $M_S^{\circ}(k) = \pm M_S(-k-1)$. In particular, $M_S(k)$ and $M_S^{\circ}(k)$ are quasi-polynomials with the same period.*

3 Antimagic Labelings

By definition of a partially magic labeling of a graph G over a subset $S \subseteq V$, all the vertices of S have the same sum $s(v)$. By letting S range over all subsets of V of size ≥ 2 , we can write the number $A_G(k)$ of weak antimagic k -labelings as an inclusion-exclusion combination of the number of positive k -partially magic labelings:

$$A_G(k) = \sum_{\substack{S \subseteq V \\ |S| \geq 2}} c_S M_S^{\circ}(k) \tag{3.1}$$

for some $c_S \in \mathbb{Z}$. Thus [Theorem 4](#) and [Corollary 8](#) imply [Theorem 2](#).

In preparation for our proof of [Theorem 3](#), we give a few basic properties of $A_G(k)$.

Lemma 9. *The quasi-polynomial $A_G(k)$ is constant zero if and only if G has a K_2 component. In either case, $A_G(0) = 0$.*

Proof. If G has a K_2 component, then clearly there is no antimagic labeling on G and so $A_G(k) = 0$. Conversely, if G is a graph with minimum number of edges such that removing any arbitrary edge results in a K_2 component in G , then each component of G is a path with 3 vertices and 2 edges, which admits an antimagic edge labeling. Now assume that G is a graph consisting of an edge e such that the graph $G \setminus e$, obtained from G by removing e , does not have any K_2 component. By induction, $G \setminus e$ admits an antimagic labeling. For the graph G , we can label the edge $e = vu$ sufficiently large such that $s(v)$ and $s(u)$ are different from each other vertex sum. Thus $A_G(k) \neq 0$.

The second statement follows from [\(3.1\)](#), since by definition $\mathcal{P}_S \subseteq [0, 1]^E$ and so $M_S^\circ(0) = L_{\mathcal{P}_S^\circ}(1) = 0$. \square

Proof of [Theorem 3](#). The second statement can be proven similarly to the first statement of [Theorem 3](#) in [Section 1](#). By [Theorem 2](#), we know that $A_G(k)$ is a quasi-polynomial in k of period ≤ 2 , and we also know that $A_G(k+1) \geq A_G(k)$. So both even and odd constituents are polynomials in k with degree at most $|E|$ and so they can have at most $|E|$ integer roots. By [Lemma 9](#), one of the roots is 0. Therefore $A_G(2|E|) > 0$. \square

4 Concluding Remarks and Open Problems

Among the more recent results on antimagic graphs are some for **directed graphs** (for which one of the endpoints of each edge e is designated to be the head, the other the tail of e); given an edge labeling of a directed graph, we denote the **oriented sum** $s(v)$ at the vertex v to be the sum of the labels of all edges oriented away from v minus the sum of the labels of all edges oriented towards v . Such a labeling is **antimagic** if each label is a distinct element of $\{1, 2, \dots, |E|\}$ and the oriented sums $s(v)$ are pairwise distinct. It is known that every directed graph whose underlying undirected graph is dense (in the sense that the minimum degree at least $C \log |V|$ for some absolute constant $C > 0$) is antimagic, and that almost every regular graph admits an orientation that is antimagic [\[12\]](#). Hefetz, Mütze, and Schwartz suggest a directed version of the Antimagic Graph Conjecture; the two natural exceptions are the complete graph K_3 on three vertices with an edge orientation that makes an oriented cycle, and $K_{1,2}$, the bipartite graph on the vertex partition $\{v_1\}$ and $\{v_2, v_3\}$ where the orientations are from v_2 to v_1 and v_1 to v_3 .

Directed Antimagic Graph Conjecture. *Every connected directed graph except for the directed graphs K_3 and $K_{1,2}$ admits an antimagic labeling.*

It is tempting to adjust our techniques to the directed settings, but there seem to be road blocks. For starters, no directed graph has a **magic labeling**, i.e., all sums $s(v)$ are equal. To see this, let A be the square matrix with A_{ij} the oriented sum of the vertex v_i using the labels of all edges between v_i and v_j . Now if L is a magic labeling with index r , the sum of each row of A equals r , and so r is an eigenvalue of A (with eigenvector $[1, 1, \dots, 1]$). However, A is by construction a skew matrix, and so it cannot have a real eigenvalue.

At any rate, a directed graph will have *partially* magic labelings, defined analogously to the undirected case, and so we can enumerate antimagic labelings according to the directed analogue of (3.1). To assert the existence of an antimagic labeling, one would like to bound the period of the antimagic quasi-polynomial, as in [Theorem 4](#). However, this does not seem possible. Namely, if the subset $S \subset V$ includes a directed path $\dots \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_s \rightarrow \dots$ such that the vertices v_2, \dots, v_{s-1} are not adjacent to any other vertices, then a completely fundamental partially magic labeling L_S with index ≥ 1 implies that the label on each edge of the path is greater than that on the previous one. Thus, contrary to the situation in [Lemma 7](#), the upper bound for the labels in L_S can be arbitrarily large. Consequently, the periods of the partial-magic quasi-polynomials, and thus those of the antimagic quasi-polynomials, can be arbitrarily large.

The papers [4, 12] give several further open problems on antimagic graphs, some of which could be tackled with the methods presented here. We close with an open problem about a natural extension of our antimagic counting function. Namely, it follows from [4] that the number of antimagic labelings of a given graph G with *distinct* labels between 1 and k is a quasi-polynomial in k . Can anything substantial be said about its period? It is unclear to us whether the methods presented here are of any help, however, any positive result would open the door to applying these ideas once more towards the Antimagic Graph Conjecture.

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